The Foundation of Probability Theory

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Probabilistic Experiment

- A *Probabilistic Experiment* is a situation in which
  - More than one thing can happen
  - The outcome is potentially uncertain

The Sample Space

- The *Sample Space* Ω of a probabilistic experiment E is the set of all possible outcomes of E.

The Sample Space

Examples:

E_1 = Toss a coin, observe whether it is a Head (H) or a Tail (T)

$\Omega_1 = \{H, T\}$

The Sample Space

Examples:

E_2 = Toss a fair die, observe the outcome.

$\Omega_2 = \{1, 2, 3, 4, 5, 6\}$

E_3 = Toss a fair coin 5 times, observe the number of heads.

Ω_3 = ? (C.P.)

The Sample Space

Examples:

E_4 = Toss a fair coin 5 times, observe the sequence of heads and tails.

Ω_4 ={HHHHH, HHHHT, HHHTH, HHHTT, HHTHH, HHTHT, HHTTH, HHTTT, ….

Even with very simple situations, the Sample Space can be quite large. Note that more than one *Probabilistic Experiment* may be defined on the same physical process.

Elementary Events vs. Compound Events

- The *Elementary Events* in a Sample Space are the finest possible partition of the sample space.
- *Compound Events* are the union of elementary events.

Elementary Events vs. Compound Events

Example:

Toss a fair die. (E2)

The *elementary events* are 1,2,3,4,5 and 6.

The events "Even" = {2,4,6}, "Odd" = {1,3,5} are examples of *compound events*.

The Axioms of Relative Frequency

Event	Relative Freq.
6	1/6
5	1/6
4	1/6
3	1/6
2	1/6
1	1/6
(>3)	3/6
Even	3/6
Odd	3/6
Even U (>3)	4/6
Even U Odd	1

Axioms of Relative Frequency

- For any events in Ω, the following facts about the relative frequencies can be established.

$$(1)\ \ 0 \le r(A) \le 1$$

$$(2)\ \ r(\Omega) = 1$$

$$(3)\ \ \text{If } A \cap B = \varnothing, \text{ then}$$

$$r(A \cup B) = r(A) + r(B)$$

Axioms of Discrete Probability

Given a probabilistic experiment E with sample space Ω and events A_i, …. The probabilities $\Pr(A_i)$ of the events are numbers satisfying the following 3 axioms:

$$\Pr(A_i) \geq 0$$

$$\Pr(\Omega) = 1$$

$$\text{If } A \cup B = \varnothing, \text{ then}$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B)$$

3 Fundamental Theorems of Probability

- *Theorem 1.* $\Pr(\varnothing) = 0$
- *Proof.* For all events *A*, $A \cap \varnothing = \varnothing$.

So the 3rd axiom applies, and we have

$$\Pr(A \cup \varnothing) = \Pr(A) + \Pr(\varnothing) = \Pr(A)$$

But, for any set *A*, $A \cup \varnothing = A$, so by subtraction, we have the result.

3 Fundamental Theorems of Probability

- *Theorem 2.* $\Pr(A) = 1 - \Pr(\bar{A})$
- *Proof.* For all events A, .

$$A \cap \bar{A} = \varnothing, \text{ so } \Pr(A \cup \bar{A}) = \Pr(A) + \Pr(\bar{A})$$

But, for any set *A*, $A \cup \bar{A} = \Omega$, so

$$\Pr(\Omega) = 1 = \Pr(A) + \Pr(\bar{A})$$

The result then follows by subtraction.

3 Fundamental Theorems of Probability

- *Theorem 3.* $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
- *Proof.* Someone from the class will prove this well known result.